

Note

A Short Proof of the Restricted Ramsey Theorem for Finite Set Systems

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We give a short proof of the restricted Ramsey theorem for finite set systems due to J. Nešetřil and V. Rödl (*J. Combin. Theory Ser. A* 34 (1983), 183–201). © 1989 Academic Press, Inc.

1. INTRODUCTION

Let $\alpha = (\alpha_1, \dots, \alpha_a)$ be a sequence of positive integers. An α -graph (resp. a set system of type α , resp., a hypergraph of type α) is a pair (X, \mathcal{E}) , where

X is a nonempty set, the set of vertices,

$\mathcal{E}: [X]^i \rightarrow \alpha_i$ for $0 < i \leq a$ is the edge-mapping.

(As usual, $[X]^i$ denotes the set of i -element subsets of X and positive integers are identified with the sets of their predecessors, e.g., $k = \{0, \dots, k-1\}$, which is the ordinal notation).

The intended interpretation is that $\mathcal{E}(A)$ gives the multiplicity of the edge A . In particular, $\mathcal{E}(A) = 0$ indicates the absence of the edge A . Note that ordinary graphs are just $(1, 2)$ -graphs in the sense defined above.

Additionally, we always assume that vertex sets are ordered by some total order which is fixed but, for convenience, never mentioned explicitly. So, actually, we consider *ordered* α -graphs and this is the usual concept with respect to Ramsey type results. For a discussion compare [2, 3].

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An *embedding* between α -graphs (X, \mathcal{E}) and $(\hat{X}, \hat{\mathcal{E}})$ is given by an order preserving injection $f: X \rightarrow \hat{X}$ satisfying

$$\mathcal{E}(A) = \hat{\mathcal{E}}(f(A)) \quad \text{for all } A \in [X]^{\leq a}.$$

For $Y \subseteq X$ the α -graph spanned by Y is $(Y, \mathcal{E} \upharpoonright [Y]^{\leq a})$.

Capital letters F, G, H denote α -graphs. The binomial coefficient $\binom{G}{H}$ denotes the set of subgraphs of G which are isomorphic to H .

An α -graph (X, \mathcal{E}) is *irreducible* if for any two distinct vertices $y, z \in X$ there exists $A \in [X]^{\leq a}$ containing y and z and such that $\mathcal{E}(A) > 0$. Intuitively: any two vertices are joined by an edge. E.g., with respect to ordinary graphs cliques are the only irreducible ones.

If \mathcal{F} is a set of α -graphs we denote by $\text{Forb}(\mathcal{F})$ the set of all finite α -graphs not containing any subgraph isomorphic to a member of \mathcal{F} , i.e.,

$$\text{Forb}(\mathcal{F}) = \left\{ G \mid G \text{ is } \alpha\text{-graph and } \binom{G}{H} = \emptyset \text{ for every } H \in \mathcal{F} \right\}.$$

The following Ramsey theorem has been proved in Nešetřil and Rödl [2, 3].

THEOREM (restricted Ramsey theorem for α -graphs). *Let \mathcal{F} be a set of irreducible α -graphs. Then, for all $G, H \in \text{Forb}(\mathcal{F})$ and every positive integer r there exists an $F \in \text{Forb}(\mathcal{F})$ satisfying $F \rightarrow (G)_r^H$, meaning that for every r -coloring $\Delta: \binom{F}{H} \rightarrow r$ there exists some $\tilde{G} \in \binom{F}{G}$ such that $\Delta \upharpoonright \binom{\tilde{G}}{H}$ is a constant coloring.*

With respect to $\mathcal{F} = \emptyset$ this has been proved, independently, in Abramson and Harrington [1]. The original proofs are quite involved and conceptually not that easy to understand, even in the case of ordinary graphs. It is the aim of this paper to give a short and simple proof for the restricted Ramsey theorem for α -graphs. This applies especially to ordinary graphs.

In Section 2 we explain the concept of left-rectified partite α -graphs and prove a simple result about these graphs. In Section 3 we define the \ast_r -amalgamation of left-rectified partite α -graphs. In Section 4 we prove the restricted Ramsey theorem with respect to $\mathcal{F} = \emptyset$ and Section 5 contains a proof of the final result.

2. LEFT-RECTIFIED PARTITE α -GRAPHS

A pair $((X_v)_{v < m}, \mathcal{E})$ is an m -partite α -graph if $(\bigcup_{v < m} X_v, \mathcal{E})$ is an α -graph such that $\mathcal{E}(A) = \emptyset$ whenever $|A \cap X_v| \geq 2$ for some $v < m$. We

always assume that the sets X_v are nonempty and mutually disjoint. The tacit order on X , then, is such that $X_0 < X_1 < \dots < X_{m-1}$.

The sets X_v are the *coordinates* of $((X_v)_{v < m}, \mathcal{E})$. For $A \subseteq \bigcup_{v < m} X_v$ we let $\text{sh}(A) = \{v < m \mid A \cap X_v \neq \emptyset\}$ be the *shadow* of A . We say that A is *crossing* if $|\text{sh}(A)| = |A|$. In particular, the m -partite α -graph $((X_v)_{v < m}, \mathcal{E})$ itself is crossing if $|X_v| = 1$ for every $v < m$.

Observe that every α -graph (X, \mathcal{E}) can be viewed as a crossing $|X|$ -partite α -graph. In particular, every irreducible α -graph is necessarily crossing.

The m -partite α -graph $((X_v)_{v < m}, \mathcal{E})$ is *left-rectified* if $\mathcal{E}(A) = \mathcal{E}(B)$ whenever $\text{sh}(A) = \text{sh}(B)$ and $\max A = \max B$, i.e., $A \cap X_{\max \text{sh}(A)} = B \cap X_{\max \text{sh}(B)}$, according to our assumption that the tacit order satisfies $X_0 < X_1 < \dots < X_{m-1}$. Note also that every crossing graph is, trivially, left-rectified.

An *embedding* between partite α -graphs $((X_v)_{v < m}, \mathcal{E})$ and $((\tilde{X}_v)_{v < \tilde{m}}, \tilde{\mathcal{E}})$ is given by an order preserving injection $f: m \rightarrow \tilde{m}$ together with order preserving injections $f: X_v \rightarrow \tilde{X}_{f(v)}$ such that

$$\mathcal{E}(A) = \tilde{\mathcal{E}}(f(A)) \quad \text{for all } A \in \left[\bigcup_{v < m} X_v \right]^{\leq a}.$$

So the additional requirement is that coordinates are preserved. Observe that partite subgraphs of left-rectified graphs are again left-rectified.

Extending our previous convention, capital letters F, G, H also denote partite α -graphs and the binomial coefficient $\binom{G}{H}_{\text{part}}$ denotes the set of all partite H -subgraphs of G .

Note that for crossing graphs G we have that $\binom{G}{H}_{\text{part}} = \binom{G}{H}$ and $\binom{F}{G}_{\text{part}} \subseteq \binom{F}{G}$ in general. We use the Ramsey arrow $F \rightarrow_{\text{part}} (G)_r^H$ in its obvious meaning, viz., for every r -coloring $\Delta: \binom{F}{H}_{\text{part}} \rightarrow r$ there exists $\tilde{G} \in \binom{F}{G}_{\text{part}}$ such that $\Delta \upharpoonright \binom{\tilde{G}}{H}_{\text{part}}$ is a constant coloring.

With respect to forbidden subgraphs we use the following notation: For m -partite α -graphs G let $\text{Nonsub}(G)$ be the set of all irreducible α -graphs H such that $\binom{G}{H} = \emptyset$.

PARTITE LEMMA A. *Let G and H be left-rectified m -partite α -graphs with H being crossing and let r be a positive integer. Then there exists a left-rectified m -partite α -graph F satisfying $F \rightarrow_{\text{part}} (G)_r^H$ and, additionally, $\text{Nonsub}(F) = \text{Nonsub}(G)$. Moreover, every vertex of F belongs to some partite G -subgraph.*

Proof of Partite Lemma A. We proceed by induction on m . For $m = 1$ the statement reduces to the pigeon hole principle. We prove it for $m + 1$. Let $G = ((X_v)_{v < m+1}, \mathcal{E})$ and $H = (m+1, \mathcal{H})$ be $(m+1)$ -partite left-rectified α -graphs, where H is crossing, and let r be a positive integer. As G

is left-rectified, every vertex $x \in X_m$ induces an $(\alpha_2, \dots, \alpha_a)$ -graph \mathcal{E}^x on m by $\mathcal{E}^x(\text{sh}(A)) = \mathcal{E}(A \cup \{x\})$ for every crossing subset $A \subseteq \bigcup_{v < m} X_v$. Let the $(\alpha_2, \dots, \alpha_a)$ -graph \mathcal{H}^m be defined analogously.

A vertex $x \in X_m$ belongs to some partite H -subgraph if and only if $\mathcal{H}^m = \mathcal{E}^x$ (without loss of generality we can assume that $(\frac{G}{H})_{\text{part}} \neq \emptyset$). Say, the set $\{x \in X_m \mid \mathcal{E}^x = \mathcal{H}^m\}$ has z elements. Let $z^* = r \cdot (z - 1) + 1$.

By G' (resp. H') we denote the left-rectified m -partite subgraphs which are spanned by the first m coordinates.

By induction, then, there exists a left-rectified m -partite α -graph F' satisfying $F' \rightarrow_{\text{part}} (G')_{\text{part}}^{H'}$ and $\text{Nonsub}(F') = \text{Nonsub}(G')$ and such that every vertex of F' belongs to some partite G' -subgraph.

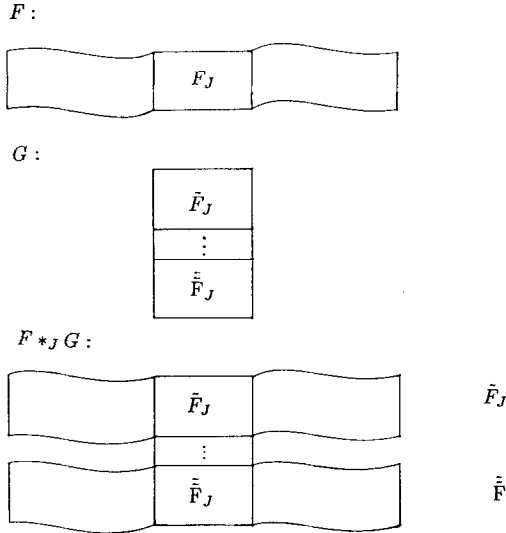
We extend F' to a left-rectified $(m+1)$ -partite α -graph by adding a largest coordinate as follows: first, we add z^* many vertices y_0, \dots, y_{z^*-1} each inducing an $(\alpha_2, \dots, \alpha_a)$ -graph \mathcal{H}^m ; second, we add vertices \hat{x} for $x \in X_m$ with $\mathcal{E}^x \neq \mathcal{H}^m$ in such a way that every z -element subset of y_0, \dots, y_{z^*-1} can be extended to a copy of X_m . Recall that we have to define an ordered set. We call the resulting graph F and claim that F has the desired properties. Obviously $\text{Nonsub}(F) = \text{Nonsub}(G)$, as $\text{Nonsub}(F') = \text{Nonsub}(G')$ and all the added vertices induce structures which already existed in G . Moreover, every G' -subgraph of F' extends to some G -subgraph of F and also every vertex from the last coordinate belongs to some G -subgraph.

So, finally, let $\Delta: (\frac{F}{H})_{\text{part}} \rightarrow r$ be an r -coloring. This induces an r^{z^*} -coloring $\Delta^*: (\frac{F'}{H'})_{\text{part}} \rightarrow r^{z^*}$ by $\Delta^*(\tilde{H}') = \Delta(\tilde{H}' \cup \{y_i \mid i < z^*\})$. Let $\tilde{G}' \in (\frac{F'}{G'})_{\text{part}}$ be monochromatic with respect to Δ^* . This induces an r -coloring of the vertices $\{y_0, \dots, y_{z^*-1}\}$ and by choice of z^* there exist z many in the same color. So extend \tilde{G} with such z vertices plus the corresponding \hat{x} -vertices to a monochromatic G -subgraph, now monochromatic with respect to Δ . ■

3. $*$ $_J$ -AMALGAMATION

Let $F = ((X_v)_{v < m}, \mathcal{F})$ be a left-rectified m -partite α -graph and let $J \subseteq m$ be a nonempty subset. By F_J we denote the subgraph of F which is spanned by the coordinates $X_j, j \in J$. Additionally let $G = ((Y_v)_{v \in J}, \mathcal{G})$ be a left-rectified $|J|$ -partite α -graph. We define the *amalgamation* $F *_J G$ of F with G along the coordinates $j \in J$ (which, then, again is a left-rectified m -partite α -graph) as follows: The subgraph of the amalgamation which is spanned by the coordinates $j \in J$ is precisely G , i.e., $(F *_J G)_J = G$. Moreover, every $\tilde{F}_J \in (\frac{G}{F_J})$ is extended to an m -partite α -graph isomorphic to F and having shadow m and, up to intersections in G , these extensions are mutually disjoint. Eventually we add edges (as few as possible) to obtain a left-

rectified graph. We hesitate to give a formal definition, but rather illustrate the construction using a picture.



The following two properties are of importance:

PROPERTY 1. Assume that every vertex of G belongs to some F_J -subgraph and that $\text{Nonsub}(G) = \text{Nonsub}(F_J)$. Then also $\text{Nonsub}(F) = \text{Nonsub}(F *_J G)$.

Proof of Property 1. By definition, every $\tilde{F}_J \in \binom{G}{F_J}$ extends to some $\tilde{F} \in \binom{F *_J G}{F_J}$. By abuse of language we denote by \tilde{F} the F -subgraph of $F *_J G$ corresponding to $\tilde{F}_J \in \binom{G}{F_J}$.

We show that for every crossing subset \mathcal{C} of $F *_J G$ there exists some $\tilde{F}_J \in \binom{G}{F_J}$ and there exists a crossing subset $\tilde{\mathcal{C}}$ of \tilde{F} such that $\text{sh}(\mathcal{C}) = \text{sh}(\tilde{\mathcal{C}})$ and both \mathcal{C} and $\tilde{\mathcal{C}}$ span isomorphic graphs. We proceed by induction on $|\mathcal{C}|$. For $\mathcal{C} = \emptyset$ the assertion holds vacuously. So let $\mathcal{C} = \mathcal{C}' \cup \{z\}$ be crossing with $\max \mathcal{C} = z$.

By induction we can assume that \mathcal{C}' already belongs to some \tilde{F} . As every vertex of G belongs to some F_J -subgraph it follows that, in particular, the vertex z belongs to \tilde{F} for some $\tilde{F}_J \in \binom{G}{F_J}$. Let $\tilde{\mathcal{C}}'$ be the corresponding copy of \mathcal{C}' belonging to \tilde{F} . As the whole graph is left-rectified we conclude that z is joined with the vertices in $\tilde{\mathcal{C}}'$ exactly as it is joined with \mathcal{C}' . Hence, $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}' \cup \{z\}$ has the desired properties. ■

PROPERTY 2. Let, additionally, H be a $|J|$ -partite crossing α -graph such that $G \rightarrow_{\text{part}} (F_J)_r^H$. Then for every r -coloring $\Delta: \binom{F *_J G}{H}_{\text{part}} \rightarrow r$ there exists an

$\tilde{F} \in ({}^F *_F^I G)_{\text{part}}$ such that all H -subgraphs of \tilde{F} having shadow J are colored the same.

Proof of Property 2. Obvious. ■

4. PARTITE LEMMA B

In this section we prove the

PARTITE LEMMA B. *Let G and H be partite left-rectified α -graphs with H being crossing and let r be a positive integer. Then there exists a partite left-rectified graph F satisfying $F \rightarrow_{\text{part}} (G)_r^H$.*

Proof of Partite Lemma B. Say, G is m -partite and H is k -partite. According to Ramsey's theorem let n be a positive integer satisfying $n \rightarrow (m)_r^k$.

Let F_0 be a left-rectified n -partite α -graph such that for every $J \in [n]^m$ there exists a G -subgraph in F_0 having shadow J . Such an F_0 can be obtained straightforwardly by placing the required G -subgraphs vertex disjointly and, eventually, adding edges to make it left-rectified. Let $(J_i)_{i < q}$ be an enumeration of $[n]^k$, the k -element subsets of n , and assume that F_i has been constructed. Let $F_i^* \rightarrow_{\text{part}} ((F_i)_{J_i})_r^H$, where $(F_i)_{J_i}$ denotes the subgraph of F_i which is spanned by the coordinates $j \in J_i$. Such an F_i^* exists by Partite Lemma A.

Then let $F_{i+1} = F_i *_{J_i} F_i^*$. We claim that $F_q \rightarrow_{\text{part}} (G)_r^H$. So let $\Delta: ({}^{F_q}_H)_{\text{part}} \rightarrow r$ be an r -coloring.

First, backtracking the construction of F_0, F_1, \dots, F_q and taking into account Property 2 of the $*_{\mathcal{F}}$ -amalgamation there exist an F_0 -subgraph $\tilde{F}_0 \in ({}^{F_0}_{r_0})_{\text{part}}$ such that $\Delta(\tilde{H}) = \Delta(\tilde{\tilde{H}})$ for all H -subgraphs $\tilde{H}, \tilde{\tilde{H}} \in ({}^{F_0}_H)_{\text{part}}$ sharing the same shadow (i.e., $\text{sh}(\tilde{H}) = \text{sh}(\tilde{\tilde{H}})$).

This induces a coloring $\Delta^*: [n]^k \rightarrow r$ by $\Delta^*(\text{sh } \tilde{H}) = \Delta(\tilde{H})$ for $\tilde{H} \in ({}^{F_0}_H)_{\text{part}}$.

By choice of n there exists $J \in [n]^m$ such that $\Delta^* \upharpoonright [J]^k$ is a constant coloring and by construction of F_0 there exists $\tilde{G} \in ({}^{F_0}_G)_{\text{part}}$ with $\text{sh}(G) = J$. In particular, then, $\Delta \upharpoonright ({}^{\tilde{G}}_H)_{\text{part}}$ is constant, as desired. ■

Remark. As a corollary, the Partite Lemma B immediately implies the restricted Ramsey theorem for α -graphs with respect to $\mathcal{F} = \emptyset$. Simply recall that every α -graph can be viewed as a crossing partite α -graph and crossing graphs are trivially left-rectified. Moreover, if G and H are crossing then

$$F \rightarrow_{\text{part}} (G)_r^H \text{ implies } F \rightarrow (G)_r^H, \text{ as } \binom{G}{H}_{\text{part}} = \binom{G}{H} \text{ and } \binom{F}{G}_{\text{part}} \subseteq \binom{F}{G}.$$

5. PROOF OF THE RESTRICTED RAMSEY THEOREM FOR α -GRAPHS

Despite of some technical differences the proof follows the pattern of the proof of the Partite Lemma B. The Partite Lemma B itself is used in so far as the Restricted Ramsey Theorem for $\mathcal{F} = \emptyset$ is needed, compare the above remarks.

Let \mathcal{F} , G , H , and r be as in the theorem. According to the case $\mathcal{F} = \emptyset$ let \tilde{F} be an α -graph satisfying $\tilde{F} \rightarrow (G)_r^H$. Say, $\tilde{F} = (n, \mathcal{E})$.

We construct a left-rectified n -partite α -graph $F_0 = ((X_v)_{v < n}, \mathcal{F}_0)$ as follows: for every $\tilde{G} \in (\tilde{F}_G)$, say, with vertex set $J_{\tilde{G}} \subseteq n$ let x_G^j , $j \in J_{\tilde{G}}$ be mutually distinct vertices. Then put $X_j = \{x_G^j \mid \tilde{G} \in (\tilde{F}_G)\}$ and define \mathcal{F}_0 by

$$\mathcal{F}_0(A) = \mathcal{E}(\text{sh}(A)) \text{ for every crossing subset } A \subseteq \bigcup_{v < n} X_v$$

$$\text{with } \max A = x_G^j \text{ and satisfying } \text{sh}(A) \subseteq J_{\tilde{G}},$$

and

$$\mathcal{F}_0(A) = 0 \quad \text{otherwise.}$$

In human language: for every $\tilde{G} \in (\tilde{F}_G)$ the α -graph F_0 contains a crossing G -subgraph having shadow $J_{\tilde{G}}$, all these G -subgraphs are mutually disjoint and, eventually, we add as few edges as possible making F_0 left-rectified.

We claim that $F_0 \in \text{Forb}(\mathcal{F})$. Assume to the contrary that that F_0 contains a crossing subset C spanning an irreducible subgraph isomorphic to some member of \mathcal{F} . Say, x_G^j is the largest vertex in C . In particular, for each $x \in C \setminus \{x_G^j\}$ there exists some $A \subseteq C$ with $\{x, x_G^j\} \subseteq A$ and $\mathcal{F}_0(A) > 0$. This implies that $\text{sh}(A) \subseteq J_{\tilde{G}}$, hence $\text{sh}(C) \subseteq J_{\tilde{G}}$. Then, again by definition of \mathcal{F}_0 the subgraph of F_0 spanned by C is isomorphic to the subgraph of \tilde{G} spanned by $\text{sh}(C)$, contradicting that $G \in \text{Forb}(\mathcal{F})$.

Now let $(J_i)_{i < q}$ be an enumeration of the vertex sets of H -subgraphs of \tilde{F} and assume that $F_i \in \text{Forb}(\mathcal{F})$ has been constructed.

Let F_i^* be a left-rectified $|J_i|$ -partite α -graph satisfying $F_i^* \rightarrow_{\text{part}} ((F_i)_{J_i})_r^H$, $\text{Nonsub}(F_i^*) = \text{Nonsub}((F_i)_{J_i})$ and every vertex of F_i^* belongs to some $(F_i)_{J_i}$ -subgraph. Such an F_i^* exists by the Partite Lemma A. Then let $F_{i+1} = F_i *_{J_i} F_i^*$. By Property 1 we know that $F_{i+1} \in \text{Forb}(\mathcal{F})$. In particular, $F_q \in \text{Forb}(\mathcal{F})$. We claim that $F_q \rightarrow (G)_r^H$, and this follows again by backtracking the construction of F_0, F_1, \dots, F_q taking into account Property 2.

Namely, given $\Delta: (\tilde{F}_H)_{\text{part}} \rightarrow r$ there exists an $\tilde{F}_0 \in (\tilde{F}_G^q)$ such that $\Delta(\tilde{H}) = \Delta(\tilde{H})$ for all H -subgraphs \tilde{H} , $\tilde{H} \in (\tilde{F}_H)_{\text{part}}$ sharing the same shadow. This induces an r -coloring $\Delta^*: (\tilde{F}_H) \rightarrow r$ by $\Delta^*(\text{sh}(\tilde{H})) = \Delta(\tilde{H})$ for $\tilde{H} \in (\tilde{F}_H)_{\text{part}}$. By choice of \tilde{F} there exists a monochromatic $\tilde{G} \in (\tilde{F}_G)$ and, finally, by construction of F_0 , there exists a crossing G -subgraph $\tilde{G}_0 \in (\tilde{F}_G)$ having shadow $J_{\tilde{G}}$. In particular, $\Delta \upharpoonright (\tilde{G}_0^H)$ is a constant coloring. ■

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